

A matrix description of weakly bipartitive and bipartitive families

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Abstract

The notions of weakly bipartitive and bipartitive families were introduced by Montgolfier (2003) as a general tool for studying some decomposition of graphs and other combinatorial structures. In this paper, we give a matrix description of these notions.

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1. Introduction

Modular decomposition has arisen as a technique that applies to many combinatorial structures such as graphs, tournaments, 2-structures, hypergraphs, and matroids, among others. It is based on module. For graphs, this notion goes back to Gallai [9]. More precisely, let $G = (V, E)$ be an undirected simple graph. A *module* of G is a set $M \subseteq V$ such that for all $x \in V \setminus M$ either $N_G(x) \cap M = \emptyset$ or $M \subseteq N_G(x)$, where $N_G(x)$ is the *neighborhood* of x , that is, $N_G(x) := \{y \in V : \{x, y\} \in E\}$. For tournaments, the notion of module can be defined in a similar way. Recall that a *tournament* is a directed graph such that for every distinct vertices x and y , either $x \rightarrow y$ or $y \rightarrow x$ and never both. Let T be a tournament with vertex set V . The

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out-neighborhood of a vertex $x \in V$ is the set $N_T^+(x) = \{y \in V : x \longrightarrow y\}$ and the *in-neighborhood* is $N_T^-(x) = \{y \in V : y \longrightarrow x\}$. A module of T is a set $M \subseteq V$ such that for all $x \in V \setminus M$ either $N_T^+(x) \cap M = \emptyset$ or $M \subseteq N_T^+(x)$.

The split decomposition of graphs and the bi-join decomposition of graphs and of tournaments can be seen as a generalization of the modular decomposition [3] and Montgolfier [10]. Let $G = (V, E)$ be an undirected simple graph and let $\{X, Y\}$ be a bipartition of V . We say that $\{X, Y\}$ is a *split* of G if there exist $X_1 \subseteq X$ and $Y_1 \subseteq Y$ such that for all $x \in X_1$, $N_G(x) \cap Y = Y_1$ and for all $x \in X \setminus X_1$, $N_G(x) \cap Y = \emptyset$. We say that $\{X, Y\}$ is a *bi-join* of G if there exist $X_1 \subseteq X$ and $Y_1 \subseteq Y$ such that for all $x \in X_1$, $N_G(x) \cap Y = Y_1$ and for all $x \in X \setminus X_1$, $N_G(x) \cap Y = Y \setminus Y_1$. Remark that if X or Y is a module of G then $\{X, Y\}$ is both a split and a bi-join of G . The notion of bi-join can be also defined for tournaments in the following way. Let T be a tournament with vertex set V . A bipartition $\{X, Y\}$ of V is a *bi-join* of T if there exist $X_1 \subseteq X$ and $Y_1 \subseteq Y$ such that for all $x \in X_1$ (resp. $x \in X \setminus X_1$), $N_T^+(x) \cap Y = Y_1$ and $N_T^-(x) \cap Y = Y \setminus Y_1$ (resp. $N_T^+(x) \cap Y = Y \setminus Y_1$ and $N_T^-(x) \cap Y = Y_1$).

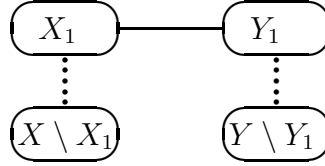


Figure 1 : A split in a graph



Figure 2 : A bi-join in a graph and in a tournament

Bipartitive families are a general tool for studying both split decomposition and bi-join decomposition. They were introduced by Montgolfier [10] as follows. Let V be a nonempty set. Two bipartitions $\{X, Y\}$ and $\{X', Y'\}$ of V *overlap* if $X \cap Y$, $X \cap Y'$, $X' \cap Y$ and $X' \cap Y'$ are nonempty. A family \mathcal{F} of bipartitions of V is *weakly bipartitive* if:

Q1) for all $v \in V$, $\{\{v\}, V \setminus \{v\}\}$ is in \mathcal{F} .

Q2) for all $\{X, Y\}$ and $\{X', Y'\}$ in \mathcal{F} such that $\{X, Y\}$ overlaps $\{X', Y'\}$, the four bipartitions $\{X \cap X', Y \cup Y'\}$, $\{X \cap Y', Y \cup X'\}$, $\{Y \cap X', X \cup Y'\}$ and $\{Y \cap Y', X \cup X'\}$ are in \mathcal{F} .

A weakly bipartitive family \mathcal{F} is *bipartitive* if it satisfies the following additional condition:

Q3) for all $\{X, Y\}$ and $\{X', Y'\}$ which overlap in \mathcal{F} , $\{X \Delta X', X \Delta Y'\}$ is in \mathcal{F} .

Cunningham [3] proved that the family of splits of a connected graph is bipartitive. The same result was obtained for the family of bi-joins of a graph by Montgolfier [10]. For tournaments, the family of bi-joins is only weakly bipartitive.

We will present now another important example of weakly bipartitive family which comes from the works of Hartfiel and Loewy [5] and of Loewy [8]. Let $A = [a_{ij}]_{1 \leq i, j \leq n}$ be a $n \times n$ matrix with entries in a field \mathbb{K} and let X, Y be two nonempty subsets of $[n]$ (where $[n] := \{1, \dots, n\}$). We denote by $A[X, Y]$ the submatrix of A having row indices in X and column indices in Y . The matrix A is *irreducible* if for any proper subset X of $[n]$, both of matrices $A[X, [n] \setminus X]$ and $A[[n] \setminus X, X]$ are nonzero. An *HL-bipartition* of A is a partition $\{X, Y\}$ of $[n]$ such that both of matrices $A[X, Y]$ and $A[Y, X]$ have rank at most 1. The concept of HL-bipartitions is equivalent to that of HL-clan [1]. In the case when A is irreducible, the family of its HL-bipartitions is weakly bipartitive (see Lemma 1 of [8]).

Splits and bi-joins can be interpreted in terms of HL-bipartitions. More precisely, we will prove in the next section that the splits (resp. the bi-joins) of an undirected simple graph G with vertex set $[n]$, are exactly the HL-bipartitions of its adjacency matrix (resp. Seidel adjacency matrix). Likewise, the bi-joins of a tournament T with vertex set $[n]$ are the HL-bipartitions of its Seidel adjacency matrix.

Throughout this paper, the family of HL-bipartitions of a matrix A is denoted by \mathcal{H}_A . Our main result is the following theorem.

Theorem 1.1. *If A is a symmetric and irreducible $n \times n$ matrix over a field \mathbb{K} then \mathcal{H}_A is bipartitive. Conversely, if \mathcal{F} is a weakly bipartitive family of $[n]$ then there exists an irreducible matrix A with entries in $\{-1, 0, 1\}$ such that $\mathcal{F} = \mathcal{H}_A$. In the particular case when \mathcal{F} is bipartitive, the matrix A can be chosen symmetric.*

2. Splits, bi-joins and HL-bipartitions

Let G be a graph with n vertices v_1, \dots, v_n . The *adjacency matrix* of G is the $n \times n$ real symmetric matrix $A(G) = [a_{ij}]_{1 \leq i, j \leq n}$ where $a_{ij} = 1$ if $\{v_i, v_j\}$ is an edge of G and $a_{ij} = 0$ otherwise. The *Seidel adjacency matrix* of G is the $n \times n$ symmetric matrix $S(G) = [s_{ij}]_{1 \leq i, j \leq n}$ in which $s_{ij} = 0$ if $i = j$ and otherwise is -1 if $\{v_i, v_j\}$ is an edge, $+1$ if it is not. The Seidel matrix was introduced by Van Lint and Seidel [11]. Adjacency matrix and Seidel matrix for a tournament are defined in the same way.

The following Proposition gives a description of splits and bi-joins in terms of HL-bipartitions.

Proposition 2.1. *Let G be a graph with vertex set $[n]$ let $\{X, Y\}$ be a bipartition of $[n]$. Then*

- i) $\{X, Y\}$ is a split of G if and only if $\{X, Y\}$ is an HL-bipartition of $A(G)$.
- ii) $\{X, Y\}$ is a bi-join of G if and only if $\{X, Y\}$ is an HL-bipartition of $S(G)$.

Proof. For positive integers r and s , we denote by $0_{r,s}$ the $r \times s$ zero matrix and by $J_{r,s}$ the $r \times s$ matrix of ones.

- i) Let $|X| := p$ and $|Y| := q$. It is easy to see that $\{X, Y\}$ is a split of G if and only if we can reorder rows and columns of $A(G)[X, Y]$ so that the resulting matrix is $0_{p,q}$, $J_{p,q}$ or one of the following matrices

$$\left(\begin{array}{c|c} J_{r,s} & 0_{r,q-s} \\ \hline 0_{p-r,s} & 0_{p-r,q-s} \end{array} \right) \quad \left(\begin{array}{c} J_{r,q} \\ \hline 0_{p-r,q} \end{array} \right) \quad \left(\begin{array}{c} J_{p,s} \\ \hline 0_{p,q-s} \end{array} \right)$$

These are the only possible forms (up to permutation of rows and columns) of a $p \times q$ $(0, 1)$ -matrices having rank at most 1.

- ii) The argument is the same as in i). It suffices to check that $\{X, Y\}$ is a bi-join of G if and only if we can reorder rows and columns of $S(G)[X, Y]$ so that the resulting matrix is $J_{p,q}$, $-J_{p,q}$ or one of the following matrices:

$$\left(\begin{array}{c|c} J_{r,s} & -J_{r,q-s} \\ \hline -J_{p-r,s} & J_{p-r,q-s} \end{array} \right) \quad \left(\begin{array}{c} J_{r,q} \\ \hline -J_{p-r,q} \end{array} \right) \quad \left(\begin{array}{c} J_{p,s} \\ \hline -J_{p,q-s} \end{array} \right)$$

□

The results of Cunningham and Montgolfier mentioned in the introduction can be deduced from the first assertion of our main theorem and the previous proposition.

A similar result of Proposition 2.1 holds for tournaments. More precisely, we have the following.

Proposition 2.2. *Let T be a tournament with vertex set $[n]$ and let $\{X, Y\}$ be a bipartition of $[n]$. Then $\{X, Y\}$ is a bi-join of T if and only if $\{X, Y\}$ is an HL-bipartition of $S(T)$.*

3. Clans of $l2$ -structures and their relationship with HL-bipartitions

Let V be a nonempty set and let $\widehat{V}^2 := \{(x, y) / x \neq y \in V\}$. Following [4] a *labelled 2-structure* on V , or a *$l2$ -structure*, for short, is a function g from \widehat{V}^2 to a set of *labels* \mathcal{C} . With each subset X of V associate the *$l2$ -substructure* $g[X]$ of g induced by X defined on X by $g[X](x, y) := g(x, y)$ for any $x \neq y \in X$. A $l2$ -structure g on a set V is *symmetric* if $g(x, y) = g(y, x)$ for every $x \neq y \in V$.

Let g be a $l2$ -structure on $[n]$ whose set of labels is a field \mathbb{K} . We associate to g the $n \times n$ matrix $M(g) = [m_{ij}]_{1 \leq i, j \leq n}$ in which $m_{ij} = 0$ if $i = j$ and $m_{ij} = g(v_i, v_j)$ otherwise. Conversely, let $A = [a_{ij}]_{1 \leq i, j \leq n}$ be a matrix with entries in a field \mathbb{K} . We associated to A the $l2$ -structure g_A on $[n]$ and set of labels \mathbb{K} such that $g_A(i, j) = a_{ij}$ for $i \neq j \in [n]$.

Given a $l2$ -structure g on V , a subset X of V is a *clan* ([4], Subsection 3.2) of g if for any $a, b \in X$ and $x \in B \setminus X$, we have $g(a, x) = g(b, x)$ and $g(x, a) = g(x, b)$.

Remark 1.

- i) Graphs and tournaments can be seen as special classes $l2$ -structure. Moreover, the notion of clan generalizes that of module.
- ii) let A be a matrix. if I is a proper clan of g_A then $\{I, [n] \setminus I\}$ is an HL-bipartition of A .

The following Proposition appears in another form in [1] (see Lemma 2.2). It describes the HL-bipartitions of a particular type of matrices called *normalized* matrices. Let $A = [a_{ij}]_{1 \leq i, j \leq n}$ be a matrix and let $v \in [n]$. We say that A is *v -normalized* if $a_{vj} = a_{jv} = 1$ for every $j \in [n] \setminus \{v\}$.

Proposition 3.1. *Let $A = [a_{ij}]_{1 \leq i, j \leq n}$ be a v -normalized matrix for some $v \in [n]$ and let $I \subseteq [n] \setminus \{h\}$. Then $\{I, [n] \setminus I\}$ is an HL-bipartition of A if and only if I is a clan of $g_A[[n] \setminus \{v\}]$.*

Proof. In order to prove the necessary condition, let $i, j \in I$ and $k \in ([n] \setminus \{v\}) \setminus I$. Since $\{I, [n] \setminus I\}$ is an HL-bipartition of A , both of matrices $A[[n] \setminus I, I]$ and $A[I, [n] \setminus I]$ have rank at most 1. It follows that $\det(A[\{v, k\}, \{i, j\}]) = \det(A[\{i, j\}, \{v, k\}]) = 0$ and so $g(k, i) = a_{ki} = a_{kj} = g(k, j)$ and $g(i, k) = a_{ik} = a_{jk} = g(j, k)$. We conclude that I is clan of $g_A([n] \setminus \{h\})$. Conversely, let I be a clan of $g_A([n] \setminus \{v\})$. Since A is v -normalized, I is a clan of g_A and then, by Remark 1, $\{I, [n] \setminus I\}$ is an HL-bipartition of A . \square

Let V be a nonempty set V and let g be a l_2 -structure on V . We denote by $Cl(g)$ the family of nonempty clans of g . This family satisfies the following well-known properties (see, for example, Subsection 3.3 of [4]).

- P1)** $V \in P$, $\emptyset \notin Cl(g)$ and for all $v \in V$, $\{v\} \in Cl(g)$;
- P2)** Given $X, Y \in Cl(g)$; if X and Y overlap, that is $X \cap Y$, $X \setminus Y$ and $Y \setminus X$ are all nonempty, then $X \cap Y \in Cl(g)$, $X \setminus Y \in Cl(g)$, $Y \setminus X \in Cl(g)$ and $X \cup Y \in Cl(g)$.

Moreover, if g is symmetric then $Cl(g)$ satisfies the additional property:

- P3)** Given $X, Y \in Cl(g)$; if X and Y overlap then $X \triangle Y = (X \setminus Y) \cup (Y \setminus X) \in Cl(g)$.

Let \mathcal{P} be a family of subsets of V . We say that \mathcal{P} is *weakly partitive* if **P1** and **P2** hold. If also **P3** holds, we say that \mathcal{P} is *partitive*. Partitive and weakly partitive families were introduced in [2]. They are closely related to partitive families as shown in the following lemma.

Lemma 3.2. *Let \mathcal{B} be a family of bipartitions of V and let $v \in V$. We denote by \mathcal{P} the family of subsets X of $V \setminus \{v\}$ such that $\{X, V \setminus X\} \in \mathcal{B}$. Then \mathcal{B} is weakly bipartitive (resp. bipartitive) if and only if \mathcal{P} is weakly partitive (resp. partitive).*

The next Theorem of gives relationship between weakly partitive family and clans family.

Theorem 3.3. *Let \mathcal{P} be a weakly partitive family on V , then there exists an l_2 -structure g on V with labels in a set of size at most 3 such that $\mathcal{P} = Cl(g)$. Moreover if \mathcal{P} is partitive family on a set V , then g can be chosen symmetric.*

The first part of this theorem was proved by Ehrenfeucht, Harju, and Rozenberg (see [4], Theorem 5.7), and later by Ille and Woodrow [6]. As noted by Ille [7], the method given in [6] can also be used to prove the second part.

4. Proof of main theorem

We start with the following result.

Proposition 4.1. *Let $A = [a_{ij}]_{1 \leq i, j \leq n}$ be an irreducible $n \times n$ matrix with entries in a field \mathbb{K} . Then for every $v \in [n]$ there is a v -normalized matrix \hat{A} with non zero entries in a field $\hat{\mathbb{K}}$ containing \mathbb{K} such that A and \hat{A} have the same HL-bipartitions. Moreover, if A is symmetric then \hat{A} can be chosen symmetric*

For the proof of this proposition, we use the following lemma.

Lemma 4.2. *Let $A = [a_{ij}]_{1 \leq i, j \leq n}$ be a irreducible matrix. Let x_1, x_2, \dots, x_n be (independent) indeterminates, $\chi = \text{diag}(x_1, x_2, \dots, x_n)$. Then we have the following statements:*

- i) *the matrix $A + \chi$ is invertible in $\mathbb{K}(x_1, x_2, \dots, x_n)$.*
- ii) *all entries of $(A + \chi)^{-1}$ are nonzero.*
- iii) *A , $A + \chi$ and $(A + \chi)^{-1}$ have the same HL-bipartitions.*

For assertions i) and ii) of this lemma, see Theorem 1 of [5]. The third assertion is a direct consequence of the following Proposition.

Proposition 4.3. [5] *Let T be an invertible matrix over \mathbb{K} , and suppose it has a block form*

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$$

where T_{11} is an invertible $k \times k$ matrix. Let $W = T^{-1}$, and partition W conformably with T , so

$$W = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix}$$

Then $\text{rank}(W_{12}) = \text{rank}(T_{12})$ and $\text{rank}(W_{21}) = \text{rank}(T_{21})$.

PROOF OF PROPOSITION 4.1. We will use the notations of Lemma 4.2. Let $(A + \chi)^{-1} := [b_{ij}]_{1 \leq i, j \leq n}$, $D := [d_i]_{1 \leq i \leq n}$ and $D' := [d'_i]_{1 \leq i \leq n}$ where $d_i = \frac{1}{b_{iv}}$, $d'_i = \frac{1}{b_{vi}}$ for $i \neq v$ and $d_v = d'_v = 1$. Clearly, the matrix $\hat{A} := D(A + X)^{-1}D'$ is v -normalized and its entries are in $\hat{\mathbb{K}} = \mathbb{K}(x_1, x_2, \dots, x_n)$. Moreover, if A is symmetric then $A + \chi$ and $(A + \chi)^{-1}$ are also symmetric. It follows that $D = D'$ and hence \hat{A} is symmetric. We conclude by applying iii) of Lemma 4.2 and the following lemma.

Lemma 4.4. *Let M be a $n \times n$ matrix and let D_1, D_2 be two $n \times n$ diagonal and invertible matrices. Then, the matrices M and D_1MD_2 have the same HL-bipartitions.*

Proof. Let X, Y be two subset of $[n]$. We have the following equalities:

$$\begin{aligned} (D_1MD_2)[X, Y] &= (D_1[X])(M[X, Y])(D_2[Y]) \\ (D_1MD_2)[Y, X] &= (D_1[Y])(M[Y, X])(D_2[X]) \end{aligned}$$

It follows $(D_1MD_2)[X, Y]$ and $(M[X, Y])$ (resp. $(D_1MD_2)[Y, X]$ and $M[Y, X]$) have the same rank because the matrices $D_1[X]$, $D_2[X]$, $D_1[Y]$ and $D_2[Y]$ are invertible. Thus, $\{X, Y\}$ is an HL-bipartition of M if and only if it is one for D_1MD_2 . \square

PROOF OF THEOREM 1.1. The fact that \mathcal{H}_A is weakly bipartitive follows from Lemma 1 of [8]. To complete the proof it suffices to check that \mathcal{H}_A satisfies the condition **Q3**. For this, let $\{X, Y\}, \{X', Y'\} \in \mathcal{H}_A$ which overlap. Then $[n] \setminus (X \cup X') = Y \cap Y' \neq \emptyset$. Let $i \in [n] \setminus (X \cup X')$. By Proposition 4.1, there is a symmetric and i -normalized matrix \hat{A} such that $\mathcal{H}_A = \mathcal{H}_{\hat{A}}$. So it suffices to prove that $\{X \Delta X', X \Delta Y'\} \in \mathcal{H}_{\hat{A}}$. By the choice of i , we have $i \notin X$ and $i \notin X'$ and then by Lemma 3.1 X and X' are clans of $g_{\hat{A}}[[n] \setminus \{i\}]$. Moreover, X and X' overlap because $\{X, Y\}, \{X', Y'\} \in \mathcal{H}_A$ overlap. Now, since \hat{A} is symmetric, $g_{\hat{A}}[[n] \setminus \{i\}]$ is symmetric and then by **P3**, $X \Delta X'$ is a clan of $g_{\hat{A}}[[n] \setminus \{i\}]$. By applying again Lemma 3.1, we deduce that $\{X \Delta X', X \Delta Y'\} \in \mathcal{H}_{\hat{A}}$.

Conversely, let \mathcal{F} be a weakly bipartitive family on a set $[n]$. We will construct an irreducible matrix A with entries in $\{-1, 0, 1\}$ such that $\mathcal{F} = \mathcal{H}_A$. From Lemma 3.2 the family $\mathcal{P} := \{X \subseteq [n-1] : \{X, [n] \setminus X\} \in \mathcal{F}\}$ is weakly partitive, then by Theorem 3.3, there exists an $l2$ -structure g on

$[n - 1]$ with labels in $\{-1, 0, 1\}$ such that $\mathcal{P} = Cl(g)$. Consider the following matrix

$$A = \left(\begin{array}{ccc|c} & & & 1 \\ & M(g) & & \vdots \\ & & & 1 \\ \hline 1 & \dots & 1 & 0 \end{array} \right)$$

Clearly, this matrix is n -normalized and then it is irreducible. To prove that $\mathcal{F} = \mathcal{H}_A$, let $\{X, [n] \setminus X\}$ be a bipartition of $[n]$ and assume for example that $n \notin X$. By Lemma 3.1, $\{X, [n] \setminus X\} \in \mathcal{H}_A$ if and only if X is a clan of g_A $[1, \dots, n - 1] = g$. Then $\{X, [n] \setminus X\} \in \mathcal{H}_A$ if and only if $X \in \mathcal{P}$ or equivalently $\{X, [n] \setminus X\} \in \mathcal{F}$ because $\mathcal{P} = Cl(g)$.

Now if \mathcal{F} is bipartitive, then the family $\mathcal{P} := \{X \subseteq [n - 1] : \{X, [n] \setminus X\} \in \mathcal{F}\}$ is partitive. By Theorem 3.3, we can choose g symmetric, which implies that A is symmetric. \square

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